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A magic connection between massive and massless diagrams

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Abstract

A useful connection between two-loop massive vacuum integrals and one-loop off-shell triangle diagrams with massless internal particles is established for arbitrary values of the space-time dimension n .

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1. Many experiments testing the Standard Model and its extensions are sensitive to the values of two-loop corrections to physical quantities and require theoretical predictions for these contributions. Since evaluating massive two-loop diagrams is a tricky business, looking for non-trivial connections between different diagrams may be of certain interest.

In a previous paper [1], we noticed that, apart from some simple logarithmic terms, the finite part of the two-loop vacuum integral with three massive propagators involves a non-trivial function of the masses, which is exactly the same as a one-loop triangle with three massless propagators whose external momenta squared are equal to those masses squared. This suggests that there is a connection between these two seemingly unrelated diagrams. The purpose of this paper is to provide an explanation of this connection (for arbitrary values of the space-time dimension) and to explore some of its consequences.

One of the important applications of the results for massive two-loop vacuum integrals is the small momentum expansion of two-loop diagrams with non-zero external momenta (see, e.g., in [1, 2]). Moreover, application of the general theory of asymptotic expansions [3] shows that analogous integrals also appear in the large momentum expansion and in the “zero-threshold” expansion (see in refs. [4, 5]). Another application is the calculation of two-loop contributions to the ρ -parameter (see in [6, 7]). Furthermore, three-loop calculations in dimensional regularization [8] (with the space-time dimension $n = 4 - 2\varepsilon$) require knowledge of the order ε contribution to two-loop vacuum diagrams. As an example, we can mention recent calculations of three-loop corrections to the ρ -parameter [9], and some other developments [10, 5].

The present paper is organized as follows. In section 2 we give definitions and different representations of the integrals considered. In section 3 we present some results for dimensionally regulated massless triangle diagrams. In section 4 we derive the connection between massive and massless integrals and use the corresponding results of section 3 to get the ε -part of two-loop vacuum integrals with different masses. In section 5 we consider what the general results yield for the important case of equal masses. In section 6 (conclusion) we discuss the main results of the paper.

2. We shall discuss the following two types of Feynman integrals (see Fig. 1a,b):

$$I(n; \nu_1, \nu_2, \nu_3 | m_1^2, m_2^2, m_3^2) \equiv \int \int \frac{d^n p \, d^n q}{(p^2 - m_1^2)^{\nu_1} (q^2 - m_2^2)^{\nu_2} ((p - q)^2 - m_3^2)^{\nu_3}}, \quad (1)$$

$$J(n; \nu_1, \nu_2, \nu_3 | p_1^2, p_2^2, p_3^2) \equiv \int \frac{d^n r}{((p_2 - r)^2)^{\nu_1} ((p_1 + r)^2)^{\nu_2} (r^2)^{\nu_3}}, \quad (2)$$



Figure 1: Two-loop vacuum diagram (a) and one-loop triangle (b)

where $p_3 = -(p_1 + p_2)$. To establish a connection between (1) and (2), we shall consider $p_i^2 = m_i^2$, and we shall use dimensionless variables

$$x \equiv p_1^2/p_3^2 = m_1^2/m_3^2, \quad y \equiv p_2^2/p_3^2 = m_2^2/m_3^2. \quad (3)$$

Below we shall omit the arguments m_i^2 and p_i^2 in the integrals I and J , respectively.

Feynman parametric representations:

$$I(n; \nu_1, \nu_2, \nu_3) = i^{2-2\Sigma\nu_i} \pi^n \frac{\Gamma(\Sigma\nu_i - n)}{\prod \Gamma(\nu_i)} \int \int \int_{\alpha_i \geq 0} \frac{\prod \alpha_i^{\nu_i-1} d\alpha_i \delta(\Sigma\alpha_i - 1)}{(\alpha_1\alpha_2 + \alpha_1\alpha_3 + \alpha_2\alpha_3)^{n/2} (\Sigma\alpha_i m_i^2)^{\Sigma\nu_i - n}}, \quad (4)$$

$$J(n; \nu_1, \nu_2, \nu_3) = i^{1-n} \pi^{n/2} \frac{\Gamma(\Sigma\nu_i - n/2)}{\prod \Gamma(\nu_i)} \int \int \int_{\alpha_i \geq 0} \frac{\prod \alpha_i^{\nu_i-1} d\alpha_i \delta(\Sigma\alpha_i - 1)}{(\alpha_1\alpha_2 p_3^2 + \alpha_1\alpha_3 p_2^2 + \alpha_2\alpha_3 p_1^2)^{\Sigma\nu_i - n/2}}, \quad (5)$$

where Σ and \prod denote the sum and the product from $i = 1$ to 3, respectively. Because of the delta functions, the integrations in (4) and (5) can be restricted to $\alpha_i \leq 1$. Nevertheless, we prefer to write them in the present form, to simplify the discussion of parameter transformations we are going to describe below.

Mellin-Barnes contour integral representations:

$$I(n; \nu_1, \nu_2, \nu_3) = \frac{\pi^n i^{2-2\Sigma\nu_i} (m_3^2)^{n-\Sigma\nu_i}}{\Gamma(n/2) \prod \Gamma(\nu_i)} \frac{1}{(2\pi i)^2} \int_{-i\infty}^{i\infty} \int_{-i\infty}^{i\infty} ds dt x^s y^t \Gamma(-s) \Gamma(-t) \\ \times \Gamma(n/2 - \nu_1 - s) \Gamma(n/2 - \nu_2 - t) \Gamma(\nu_1 + \nu_2 - n/2 + s + t) \Gamma\left(\sum \nu_i - n + s + t\right), \quad (6)$$

$$J(n; \nu_1, \nu_2, \nu_3) = \frac{\pi^{n/2} i^{1-n} (p_3^2)^{n/2-\Sigma\nu_i}}{\Gamma(n - \sum \nu_i) \prod \Gamma(\nu_i)} \frac{1}{(2\pi i)^2} \int_{-i\infty}^{i\infty} \int_{-i\infty}^{i\infty} ds dt x^s y^t \Gamma(-s) \Gamma(-t) \\ \times \Gamma(n/2 - \nu_2 - \nu_3 - s) \Gamma(n/2 - \nu_1 - \nu_3 - t) \Gamma(\nu_3 + s + t) \Gamma\left(\sum \nu_i - n/2 + s + t\right), \quad (7)$$

where the integration contours are chosen so as to separate the “right” and “left” series of poles of gamma functions in the integrand. By use of the residue theorem, the result for arbitrary n and ν_i can be found in terms of hypergeometric functions of two variables (see [11, 12, 1]).

Uniqueness condition: When the ν_i and n are related by $\nu_1 + \nu_2 + \nu_3 = n$, a very simple result can be obtained from (5) for such a “unique” triangle [13] (see also in [14]):

$$J(n; \nu_1, \nu_2, \nu_3) \Big|_{\Sigma\nu_i=n} = \pi^{n/2} i^{1-n} \prod_{i=1}^3 \frac{\Gamma(n/2 - \nu_i)}{\Gamma(\nu_i)} \frac{1}{(p_i^2)^{n/2-\nu_i}}. \quad (8)$$

3. In the paper [15], the following representation valid for arbitrary ε was obtained:

$$J(4 - 2\varepsilon; 1, 1, 1) = \frac{\pi^{2-\varepsilon} i^{1+2\varepsilon}}{(p_3^2)^{1+\varepsilon}} \frac{\Gamma(1+\varepsilon) \Gamma^2(1-\varepsilon)}{\Gamma(1-2\varepsilon)} \frac{1}{\varepsilon} \int_0^1 \frac{d\xi \xi^{-\varepsilon} ((y\xi)^{-\varepsilon} - (x/\xi)^{-\varepsilon})}{(y\xi^2 + (1-x-y)\xi + x)^{1-\varepsilon}}. \quad (9)$$

We managed to generalize this result to the case when one of the ν 's is arbitrary, as

$$J(4-2\varepsilon; 1, 1, 1+\delta) = \frac{\pi^{2-\varepsilon} i^{1+2\varepsilon}}{(p_3^2)^{1+\varepsilon+\delta}} \frac{\Gamma(1-\varepsilon) \Gamma(1-\varepsilon-\delta) \Gamma(1+\varepsilon+\delta)}{\Gamma(1+\delta) \Gamma(1-2\varepsilon-\delta)} \times \frac{1}{\varepsilon+\delta} \int_0^1 \frac{d\xi \xi^{-\varepsilon} \left((y\xi)^{-\varepsilon-\delta} - (x/\xi)^{-\varepsilon-\delta} \right)}{(y\xi^2 + (1-x-y)\xi + x)^{1-\varepsilon}}. \quad (10)$$

At $\delta = 0$, eq. (10) gives (9), whilst for $\varepsilon = 0$ we get eq. (15) of [15] (see also in [16]).

The expansion of $J(4-2\varepsilon; 1, 1, 1)$ in ε is

$$J(4-2\varepsilon; 1, 1, 1) = \pi^{2-\varepsilon} i^{1+2\varepsilon} (p_3^2)^{-1-\varepsilon} \Gamma(1+\varepsilon) \left\{ \Phi^{(1)}(x, y) + \varepsilon \Psi^{(1)}(x, y) + \mathcal{O}(\varepsilon^2) \right\}, \quad (11)$$

with

$$\Phi^{(1)}(x, y) = - \int_0^1 \frac{d\xi}{y\xi^2 + (1-x-y)\xi + x} \left(\ln \frac{y}{x} + 2 \ln \xi \right), \quad (12)$$

$$\Psi^{(1)}(x, y) = - \int_0^1 \frac{d\xi}{y\xi^2 + (1-x-y)\xi + x} \left(\ln \frac{y}{x} + 2 \ln \xi \right) \left[\ln \left(\frac{y\xi^2 + (1-x-y)\xi + x}{\xi} \right) - \frac{1}{2} \ln(xy) \right]. \quad (13)$$

Due to the symmetry of the original triangle diagram with respect to all the external legs, the following combinations are totally symmetric in p_1^2, p_2^2, p_3^2 :

$$(p_3^2)^{-1} \Phi^{(1)}(x, y) \quad \text{and} \quad (p_3^2)^{-1} \left(\Psi^{(1)}(x, y) + \frac{1}{3} \ln(xy) \Phi^{(1)}(x, y) \right). \quad (14)$$

The integrals (12)–(13) can be evaluated in terms of polylogarithms, and the results can be found in [15] (eqs. (11) and (29), respectively). The results for $\Phi^{(1)}$ were also presented in [12, 1, 16]. Similar results for the triangle function in four dimensions (involving dilogarithms) were presented in [17], while the ε -part of massive triangle diagrams was considered in [18]. Here, we present the results for $\Phi^{(1)}$ and $\Psi^{(1)}$ in a different form:

$$\Phi^{(1)}(x, y) = \frac{1}{2\lambda} \left\{ 4\text{Li}_2(1-z_1) + 4\text{Li}_2(1-z_2) + 4\text{Li}_2(1-z_3) + \ln^2 z_1 + \ln^2 z_2 + \ln^2 z_3 + 2 \ln x \ln z_1 + 2 \ln y \ln z_2 \right\}, \quad (15)$$

$$\begin{aligned} \Psi^{(1)}(x, y) = & -\frac{1}{2\lambda} \left\{ 4\text{Li}_3(1-z_1^{-1}) + 4\text{Li}_3(1-z_2^{-1}) + 4\text{Li}_3(1-z_3^{-1}) \right. \\ & - 4\text{Li}_3(1-z_1) - 4\text{Li}_3(1-z_2) - 4\text{Li}_3(1-z_3) \\ & + 4 \ln x \text{Li}_2(1-z_1) + 4 \ln y \text{Li}_2(1-z_2) \\ & \left. - \ln z_1 \ln z_2 \ln z_3 + \ln x \ln z_1 \ln(xz_1) + \ln y \ln z_2 \ln(yz_2) \right\}, \end{aligned} \quad (16)$$

where the variables z_i and λ are defined by

$$z_1 = \frac{(\lambda + x - y - 1)^2}{4y}, \quad z_2 = \frac{(\lambda + y - 1 - x)^2}{4x}, \quad z_3 = \frac{(\lambda + 1 - x - y)^2}{4xy}, \quad (17)$$

$$\lambda(x, y) = \sqrt{(1 - x - y)^2 - 4xy}. \quad (18)$$

The z_i are related to each other by $z_1 z_2 z_3 = 1$. The representations (15) and (16) are equivalent to the ones presented in [15] in the region $x + y < 1$, as can be shown by transformations of the trilogarithms and dilogarithms [19]. They are also valid in the regions $x > y + 1$ and $y > x + 1$. When λ approaches zero, the z_i go to one, and hence all terms inside the braces vanish¹. The analytic continuations of (15) and (16) into the region where $x + y > 1$ and $x - 1 < y < x + 1$, are obtained by adding some logarithmic terms to them, in such a way that the discontinuities along the branch cuts $z_i < 0$ are cancelled. Using the formulae of [19], the resulting expressions can be written as follows²:

$$\Phi^{(1)}(x, y) = \frac{2}{\sqrt{-\lambda^2}} \{ \text{Cl}_2(\theta_1) + \text{Cl}_2(\theta_2) + \text{Cl}_2(\theta_3) \} = \frac{2}{\sqrt{-\lambda^2}} \{ \text{Ls}_2(\theta_1) + \text{Ls}_2(\theta_2) + \text{Ls}_2(\theta_3) \}, \quad (19)$$

$$\Psi^{(1)}(x, y) = \ln\left(\frac{-\lambda^2}{xy}\right) \Phi^{(1)}(x, y) - \frac{2}{\sqrt{-\lambda^2}} \left\{ \text{Ls}_3(\theta_1) + \text{Ls}_3(\theta_2) + \text{Ls}_3(\theta_3) + \frac{\pi^3}{6} \right\}, \quad (20)$$

$$\theta_1 = 2 \arccos\left(\frac{1-x+y}{2\sqrt{y}}\right), \quad \theta_2 = 2 \arccos\left(\frac{1+x-y}{2\sqrt{x}}\right), \quad \theta_3 = 2 \arccos\left(\frac{-1+x+y}{2\sqrt{xy}}\right). \quad (21)$$

Note that $\theta_1 + \theta_2 + \theta_3 = 2\pi$. The log-sine integral is defined by

$$\text{LS}_N(\theta) = - \int_0^\theta d\theta \ln^{N-1} \left| 2 \sin \frac{\theta}{2} \right|, \quad (22)$$

and in particular, $\text{Ls}_2(\theta) = \text{Cl}_2(\theta)$. Eqs. (19)-(21) are valid for all x, y such that $\lambda^2 < 0$.

In the problem under consideration, the parabola defined by $\lambda(x, y) = 0$ is a special curve (see also in [20]). It consists of the three segments $\sqrt{x} + \sqrt{y} = 1$, $\sqrt{x} - \sqrt{y} = 1$ and $\sqrt{y} - \sqrt{x} = 1$. On this curve, a simple result for arbitrary space-time dimension can be obtained using eq. (9),

$$J(4 - 2\varepsilon; 1, 1, 1)|_{\lambda=0} = \frac{\pi^{2-\varepsilon} i^{1+2\varepsilon}}{(p_3^2)^{1+\varepsilon}} \frac{\Gamma(\varepsilon)\Gamma^2(1-\varepsilon)}{\Gamma(2-2\varepsilon)} \left\{ \frac{x+y-1}{2xy} + \frac{y+1-x}{2y} x^{-\varepsilon} + \frac{1+x-y}{2x} y^{-\varepsilon} \right\}, \quad (23)$$

which is clearly symmetric in p_1^2 , p_2^2 and p_3^2 and can be written as

$$- \frac{\pi^{2-\varepsilon} i^{1+2\varepsilon}}{p_1^2 p_2^2 p_3^2} \frac{\Gamma(\varepsilon)\Gamma^2(1-\varepsilon)}{\Gamma(2-2\varepsilon)} \left\{ (p_1 p_2)(p_3^2)^{1-\varepsilon} + (p_3 p_1)(p_2^2)^{1-\varepsilon} + (p_2 p_3)(p_1^2)^{1-\varepsilon} \right\}. \quad (24)$$

Expanding (23) in ε gives

$$\Phi^{(1)}(x, y)|_{\lambda=0} = -\frac{2 \ln x}{1-x+y} - \frac{2 \ln y}{1+x-y}; \quad \Psi^{(1)}(x, y)|_{\lambda=0} = \frac{\ln^2 x - 4 \ln x}{1-x+y} + \frac{\ln^2 y - 4 \ln y}{1+x-y}. \quad (25)$$

¹ In [15], the arguments of the Li_2 's and Li_3 's were chosen so as to vanish as x and y approach zero.

² Eq. (19) corresponds to (4.15) of [1]. Representations similar to (19) were also considered in [20, 21].

4. Looking at the Mellin–Barnes representations (6)–(7), it is possible to observe that

$$I(n; \nu_1, \nu_2, \nu_3) = \pi^{3n/2 - \Sigma \nu_i} i^{1-n} \frac{\Gamma(\nu_2 + \nu_3 - n/2) \Gamma(\nu_1 + \nu_3 - n/2) \Gamma(\nu_1 + \nu_2 - n/2)}{\Gamma(\nu_1) \Gamma(\nu_2) \Gamma(\nu_3)} \\ \times J(2\Sigma \nu_i - n; \nu_2 + \nu_3 - n/2, \nu_1 + \nu_3 - n/2, \nu_1 + \nu_2 - n/2). \quad (26)$$

The same relation can be obtained from the representations (4) and (5) by first inverting and then rescaling the Feynman parameters α_i ($i = 1, 2, 3$) in (4):

$$\alpha_i = \frac{1}{\alpha'_i}, \quad \alpha'_i = \mathcal{F}(\alpha''_1, \alpha''_2, \alpha''_3) \alpha''_i, \quad \mathcal{F}(\alpha''_1, \alpha''_2, \alpha''_3) = \frac{\alpha''_1{}^{-1} + \alpha''_2{}^{-1} + \alpha''_3{}^{-1}}{\alpha''_1 + \alpha''_2 + \alpha''_3}, \quad (27)$$

where the scaling factor \mathcal{F} has been chosen to restore the argument of the delta function to its original form. Due to the homogeneity of the integrand, the effect of the rescaling is to multiply it by a factor \mathcal{F}^{-3} , which is precisely cancelled by the Jacobian associated with the change of variables $\alpha'_i \rightarrow \alpha''_i$ [22]. The result has the structure of (5).

One's first impression may be that the relation (26) does not look very useful. In particular, in the case $n = 4 - 2\varepsilon$, $\nu_1 = \nu_2 = \nu_3 = 1$ it yields

$$I(4 - 2\varepsilon; 1, 1, 1) = \pi^{3-3\varepsilon} i^{1+2\varepsilon} \Gamma^3(\varepsilon) J(2 + 2\varepsilon; \varepsilon, \varepsilon, \varepsilon), \quad (28)$$

with some “strange” integral on the r.h.s. However, we can use the uniqueness relation (8) to transform the integrals J . Applying (8) with respect to all three external legs of the three-point diagram gives:

$$\Gamma(\nu_1) \Gamma(\nu_2) \Gamma(\nu_3) \Gamma(n - \Sigma \nu_i) J(n; \nu_1, \nu_2, \nu_3) \\ = \Gamma(n/2 - \nu_1) \Gamma(n/2 - \nu_2) \Gamma(n/2 - \nu_3) \Gamma(\Sigma \nu_i - n/2) \\ \times (p_1^2)^{n/2 - \nu_2 - \nu_3} (p_2^2)^{n/2 - \nu_1 - \nu_3} (p_3^2)^{n/2 - \nu_1 - \nu_2} J(n; n/2 - \nu_1, n/2 - \nu_2, n/2 - \nu_3). \quad (29)$$

Combining (26) and (29), we get

$$I(n; \nu_1, \nu_2, \nu_3) = \pi^{3n/2 - \Sigma \nu_i} i^{1-n} \left(\prod_{i=1}^3 (m_i^2)^{n/2 - \nu_i} \right) \frac{\Gamma(\Sigma \nu_i - n)}{\Gamma(n/2)} J(2\Sigma \nu_i - n; \nu_1, \nu_2, \nu_3). \quad (30)$$

Now, the powers ν_i are the same on the l.h.s. and on the r.h.s. whilst the values of the space-time dimension are different. We shall call eq. (30) a “magic” connection. It can also be derived by a change of variables in the Feynman parametric representation or in the Mellin–Barnes contour integrals.

Let us consider what the “magic” connection gives for the most interesting case $\nu_1 = \nu_2 = \nu_3 = 1$ at different values of n .

For $n = 2 - 2\varepsilon$, we get

$$I(2 - 2\varepsilon; 1, 1, 1) = \pi^{-3\varepsilon} i^{-1+2\varepsilon} (m_1^2 m_2^2 m_3^2)^{-\varepsilon} \frac{\Gamma(1 + 2\varepsilon)}{\Gamma(1 - \varepsilon)} J(4 + 2\varepsilon; 1, 1, 1). \quad (31)$$

So, the dimensionally-regularized triangle integral $J(4 + 2\varepsilon; 1, 1, 1)$ can be related to the two-dimensional integral $I(2 - 2\varepsilon; 1, 1, 1)$ (also dimensionally-regularized, but note that signs of ε are different!). Since both integrals are convergent as $\varepsilon \rightarrow 0$, we can put $\varepsilon = 0$:

$$I(2; 1, 1, 1) = -i J(4; 1, 1, 1) = \pi^2 m_3^{-2} \Phi^{(1)}(x, y). \quad (32)$$

For $n = 3 - 2\varepsilon$, we get

$$I(3 - 2\varepsilon; 1, 1, 1) = \pi^{3/2-3\varepsilon} i^{2+2\varepsilon} (m_1^2 m_2^2 m_3^2)^{1/2-\varepsilon} \frac{\Gamma(2\varepsilon)}{\Gamma(3/2 - \varepsilon)} J(3 + 2\varepsilon; 1, 1, 1). \quad (33)$$

In this case, both integrals are three-dimensional, but (again!) the signs of ε are different on the l.h.s. and on the r.h.s. Note that on the r.h.s. we also have a singular factor $\Gamma(2\varepsilon)$. So, to get the result for the singular and “constant” (in ε) terms of $I(3 - 2\varepsilon; 1, 1, 1)$, we need to know $J(3 + 2\varepsilon; 1, 1, 1)$ up to the ε -part. Using the representation (9), it can be easily calculated, and we arrive at the following result:

$$I(3 - 2\varepsilon; 1, 1, 1) = \pi^{4-2\varepsilon} \Gamma^2(1 + \varepsilon) \left\{ \frac{1}{\varepsilon} + 2 - 4 \ln(m_1 + m_2 + m_3) \right\} + \mathcal{O}(\varepsilon). \quad (34)$$

This corresponds to the result presented in [23], eq. (110). Note that simple results for three-dimensional triangles were presented in [24].

For $n = 4 - 2\varepsilon$, eq. (30) gives:

$$I(4 - 2\varepsilon; 1, 1, 1) = \pi^{3-3\varepsilon} i^{1+2\varepsilon} (m_1^2 m_2^2 m_3^2)^{1-\varepsilon} \frac{\Gamma(-1 + 2\varepsilon)}{\Gamma(2 - \varepsilon)} J(2 + 2\varepsilon; 1, 1, 1). \quad (35)$$

Using Feynman parameters, it is easy to show that

$$J(2 + 2\varepsilon; 1, 1, 1) = -\pi^{-1} \{ J(4 + 2\varepsilon; 2, 1, 1) + J(4 + 2\varepsilon; 1, 2, 1) + J(4 + 2\varepsilon; 1, 1, 2) \}. \quad (36)$$

Now, we use the formula [12] obtained by integration by parts [25],

$$\begin{aligned} J(4 + 2\varepsilon; 1, 1, 2) &= (p_1^2 p_2^2)^{-1} \left\{ -(p_1^2 + p_2^2 - p_3^2) \varepsilon J(4 + 2\varepsilon; 1, 1, 1) \right. \\ &\quad \left. + p_1^2 J(4 + 2\varepsilon; 0, 2, 1) + p_2^2 J(4 + 2\varepsilon; 2, 0, 1) - p_3^2 J(4 + 2\varepsilon; 2, 1, 0) \right\} \end{aligned} \quad (37)$$

(note that the sign of ε is different than in [12]). The integrals J with one of the ν 's equal to zero correspond to massless one-loop two-point functions. In such a way, we get

$$\begin{aligned} &\{ J(4 + 2\varepsilon; 2, 1, 1) + J(4 + 2\varepsilon; 1, 2, 1) + J(4 + 2\varepsilon; 1, 1, 2) \} \\ &= (p_1^2 p_2^2 p_3^2)^{-1} \left\{ -\varepsilon \Delta(p_1^2, p_2^2, p_3^2) J(4 + 2\varepsilon; 1, 1, 1) + p_1^2 (-p_1^2 + p_2^2 + p_3^2) J(4 + 2\varepsilon; 0, 2, 1) \right. \\ &\quad \left. + p_2^2 (p_1^2 - p_2^2 + p_3^2) J(4 + 2\varepsilon; 2, 0, 1) + p_3^2 (p_1^2 + p_2^2 - p_3^2) J(4 + 2\varepsilon; 2, 1, 0) \right\} \end{aligned} \quad (38)$$

where

$$\Delta(p_1^2, p_2^2, p_3^2) = 2p_1^2 p_2^2 + 2p_1^2 p_3^2 + 2p_2^2 p_3^2 - (p_1^2)^2 - (p_2^2)^2 - (p_3^2)^2 = -(p_3^2)^2 \lambda^2(x, y) \quad (39)$$

is connected with the Källén function.

Thus, the connection can be written in the following symmetric form:

$$\begin{aligned} I(4 - 2\varepsilon; 1, 1, 1) &= -\frac{1}{2} \pi^{4-2\varepsilon} (m_1^2 m_2^2 m_3^2)^{-\varepsilon} [(1 - \varepsilon)(1 - 2\varepsilon)]^{-1} \\ &\times \left\{ -\frac{1}{\pi^{2+\varepsilon} i^{1-2\varepsilon}} \frac{\Gamma(1 + 2\varepsilon)}{\Gamma(1 - \varepsilon)} J(4 + 2\varepsilon; 1, 1, 1) \Delta(m_1^2, m_2^2, m_3^2) \right. \\ &\quad \left. + \Gamma^2(\varepsilon) \left[(m_1^2)^\varepsilon (-m_1^2 + m_2^2 + m_3^2) + (m_2^2)^\varepsilon (m_1^2 - m_2^2 + m_3^2) + (m_3^2)^\varepsilon (m_1^2 + m_2^2 - m_3^2) \right] \right\}. \end{aligned} \quad (40)$$

Using (11) to expand in ε and keeping the terms up to the order ε , we get

$$\begin{aligned}
I(4-2\varepsilon; 1, 1, 1) &= \frac{1}{2} \pi^{4-2\varepsilon} (m_3^2)^{1-2\varepsilon} \Gamma^2(1+\varepsilon) [(1-\varepsilon)(1-2\varepsilon)]^{-1} \\
&\times \left\{ -\frac{1}{\varepsilon^2} (1+x+y) + \frac{2}{\varepsilon} (x \ln x + y \ln y) - x \ln^2 x - y \ln^2 y + (1-x-y) \ln x \ln y \right. \\
&\quad + \frac{1}{3} \varepsilon (x \ln^3 x + y \ln^3 y) - \frac{1}{2} \varepsilon (1-x-y) \ln x \ln y (\ln x + \ln y) \\
&\quad \left. - \lambda^2(x, y) (1 - \varepsilon(\ln x + \ln y)) \Phi^{(1)}(x, y) + \varepsilon \lambda^2(x, y) \Psi^{(1)}(x, y) \right\} + \mathcal{O}(\varepsilon^2). \quad (41)
\end{aligned}$$

The divergent and constant (in ε) terms coincide with the result of [1], eq. (4.9).

It could be noted that, using eqs. (40) and (31), we can write an exact relation between integrals I with different values of the space-time dimension:

$$\begin{aligned}
I(4-2\varepsilon; 1, 1, 1) &= -\frac{1}{2} \pi^{4-2\varepsilon} [(1-\varepsilon)(1-2\varepsilon)]^{-1} \left\{ -\pi^{-2+2\varepsilon} \Delta(m_1^2, m_2^2, m_3^2) I(2-2\varepsilon; 1, 1, 1) \right. \\
&\quad \left. + \Gamma^2(\varepsilon) (m_1^2 m_2^2 m_3^2)^{-\varepsilon} \left[(m_1^2)^\varepsilon (-m_1^2 + m_2^2 + m_3^2) + (m_2^2)^\varepsilon (m_1^2 - m_2^2 + m_3^2) + (m_3^2)^\varepsilon (m_1^2 + m_2^2 - m_3^2) \right] \right\}. \quad (42)
\end{aligned}$$

Using (30), an analogous relation can also be written for one-loop triangles, namely:

$$\begin{aligned}
J(2+2\varepsilon; 1, 1, 1) &= (p_1^2 p_2^2 p_3^2)^{-1} \left\{ \pi^{-1} \varepsilon \Delta(p_1^2, p_2^2, p_3^2) J(4+2\varepsilon; 1, 1, 1) \right. \\
&\quad \left. + 2\pi^{1+\varepsilon} i^{1-2\varepsilon} \frac{\Gamma(1-\varepsilon)\Gamma^2(1+\varepsilon)}{\Gamma(1+2\varepsilon)} \frac{1}{\varepsilon} \left[(p_1 p_2)(p_3^2)^\varepsilon + (p_3 p_1)(p_2^2)^\varepsilon + (p_2 p_3)(p_1^2)^\varepsilon \right] \right\}. \quad (43)
\end{aligned}$$

This is a special case ($N=3$) of an identity given in [26] relating one-loop N -point integrals to $(N-1)$ -point ones. Note that at $\lambda = 0$ ($\Delta = 0$) the first term in the braces on the r.h.s. disappears, and, changing ε into $(1-\varepsilon)$, we obtain nothing but the result (24).

5. In many realistic applications, the masses of internal particles are equal. For this case ($m_1 = m_2 = m_3 \equiv m$, $x = y = 1$), the integral representations (12) and (13) give

$$\Phi^{(1)}(1, 1) = -2 \int_0^1 \frac{d\xi}{1-\xi+\xi^2} \ln \xi, \quad \Psi^{(1)}(1, 1) = -2 \int_0^1 \frac{d\xi}{1-\xi+\xi^2} \ln \xi \ln \left(\frac{1-\xi+\xi^2}{\xi} \right). \quad (44)$$

We note that the first of these two integrals was also presented in [27].

The angles (21) are now all equal to $2\pi/3$, so that eqs. (20) and (19) are reduced to³

$$\Phi^{(1)}(1, 1) = \frac{6}{\sqrt{3}} \text{Cl}_2 \left(\frac{2\pi}{3} \right) = \frac{4}{\sqrt{3}} \text{Cl}_2 \left(\frac{\pi}{3} \right), \quad (45)$$

$$\Psi^{(1)}(1, 1) = \frac{1}{\sqrt{3}} \left\{ -6 \text{Ls}_3 \left(\frac{2\pi}{3} \right) + 4 \ln 3 \text{Cl}_2 \left(\frac{\pi}{3} \right) - \frac{1}{3} \pi^3 \right\}. \quad (46)$$

Eqs. (45)-(46) can be expressed in terms of the generalized inverse tangent integral [19],

$$\text{Ti}_N(z) = \frac{1}{2i} (\text{Li}_N(iz) - \text{Li}_N(-iz)) = \frac{(-1)^{N-1}}{(N-1)!} z \int_0^1 d\xi \frac{\ln^{N-1} \xi}{1+z^2 \xi^2}, \quad (47)$$

³ The representation in terms of $\text{Cl}_2 \left(\frac{\pi}{3} \right)$ is well-known (see e.g. in [6, 28]). $\Phi^{(1)}(1, 1)$ can be also expressed as $\Phi^{(1)}(1, 1) = \frac{2}{3} \left[\psi' \left(\frac{1}{3} \right) - \frac{2}{3} \pi^2 \right]$, see in [29].

whose Taylor series is

$$\text{Ti}_N(z) = z \sum_{j=0}^{\infty} \frac{(-z^2)^j}{(2j+1)^N} = \frac{z}{1^N} - \frac{z^3}{3^N} + \frac{z^5}{5^N} - \dots, \quad |z| < 1. \quad (48)$$

Using

$$\text{Ti}_3\left(\frac{1}{\sqrt{3}}\right) = \frac{1}{\sqrt{3}} \sum_{j=0}^{\infty} \frac{(-1)^j}{3^j (2j+1)^3} = 0.570681635... \quad (49)$$

$$\text{Cl}_2\left(\frac{\pi}{3}\right) = \frac{6}{5} \text{Ti}_2\left(\frac{1}{\sqrt{3}}\right) + \frac{1}{10} \pi \ln 3 = 1.014941606..., \quad (50)$$

$$\text{Ls}_3\left(\frac{2\pi}{3}\right) = \frac{8}{5} \text{Ti}_3\left(\frac{1}{\sqrt{3}}\right) + \frac{2}{3} \ln 3 \text{Cl}_2\left(\frac{\pi}{3}\right) - \frac{1}{30} \pi \ln^2 3 - \frac{16}{135} \pi^3 = -2.144767213... \quad (51)$$

we find

$$\Phi^{(1)}(1, 1) = \frac{2}{5\sqrt{3}} \left\{ 12 \text{Ti}_2\left(\frac{1}{\sqrt{3}}\right) + \pi \ln 3 \right\} = 2.343907239..., \quad (52)$$

$$\Psi^{(1)}(1, 1) = \frac{1}{5\sqrt{3}} \left\{ -48 \text{Ti}_3\left(\frac{1}{\sqrt{3}}\right) + \pi \ln^2 3 + \frac{17}{9} \pi^3 \right\} = 4.037576132... \quad (53)$$

6. In this paper we have derived a useful relation (30) between two *very different* types of Feynman diagrams: two-loop massive vacuum integrals (1) and one-loop three-point functions (2). This “magic” connection is valid for any values of the space-time dimension and the powers of propagators. While the powers of propagators are the same, the massive and massless integrals related by (30) have different values of the space-time dimension. Nevertheless, using some additional transformations it is possible to relate dimensionally-regulated integrals considered around $n = 4$ (see eq. (40)). However, we get different signs of ε on the l.h.s. and on the r.h.s. ($n = 4 \mp 2\varepsilon$). As a result, some ultraviolet singularities of massive integrals can correspond to infrared singularities of massless triangles. This is not dangerous, since the magic connection relates diagrams of different type and does not introduce additional mixing of different singularities in diagrams of the same type.

An important application considered in the paper is the result for the ε -part of the two-loop massive vacuum diagram with different masses, eq. (41). For the equal-mass case, a new transcendental constant is shown to appear. It is not excluded that it can be connected with an analytically-unknown constant occurring in three-loop calculations of the ρ -parameter [9] (see also in [30]).

Using integral representations (9)–(10), higher terms of the expansion in ε (and in δ) can also be obtained. Moreover, the corresponding non-trivial functions will be the same in both cases, due to the magic connection.

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